

TIS-B: CALCULATION OF NAVIGATION ACCURACY CATEGORY FOR POSITION AND VELOCITY PARAMETERS

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Abstract

Traffic Information Service - Broadcast (TIS-B) is a surveillance service that derives traffic information from one or more ground surveillance sources and broadcasts to Automatic Dependant Surveillance – Broadcast (ADS-B) equipped aircraft or surface vehicles, with the intention of supporting Aircraft Separation Assurance (ASA) applications. ASA Minimum Aviation System Performance Standards (MASPS) require that the Navigation Accuracy Category for Position (NAC_P) and velocity (NAC_V) be reported so that ASA applications may determine whether the reported position and velocity have an acceptable level of accuracy for the intended use. This paper develops two methods to compute NAC_P and NAC_V . The two methods are denoted *exact* and *bounded*, and their performance is investigated in computing the 95% containment threshold for a broad range of values of elliptical eccentricity. The algorithms for the two methods can be easily implemented regardless of the eccentricity of the underlying uncertainty.

The proposed algorithm computes NAC_P and NAC_V parameters for position and rate estimates that are derived from Kalman filter based estimators. These estimators provide a covariance matrix that is a measure of the uncertainty associated with the estimates. It is first shown that the covariance matrix computed by Kalman-based estimators can be partitioned into horizontal position and rate terms. Eigenvalues can be computed for the partitioned covariance matrix to provide the squares of the major and minor axes of the representative uncertainty ellipses regardless of their orientation with respect to the reference frame. Conversion factors from the elliptical to circular bounds were computed for relative accuracy that accounts for random error only. A follow-on paper addresses absolute accuracy that in addition to random error accounts for uncorrected bias errors that may occur after an aircraft maneuver due to a position lag in the estimated positions.

Introduction

TIS-B is a surveillance service that derives traffic information from one or more ground surveillance sources and broadcasts to ADS-B equipped aircraft or surface vehicles, with the intention of supporting ASA applications. Aircraft ASA MASPS require that the Navigation Accuracy Category for Position (NAC_P) and velocity (NAC_V) be reported so that ASA applications may determine whether the reported position and velocity have an acceptable level of accuracy for the intended use. Error statistics are needed for the calculation of NAC_P and NAC_V . Definitions of NAC_P and NAC_V are:

NAC_P : The Estimated Position Uncertainty (EPU) and Vertical Estimated Position Uncertainty (VEPU) are 95% accuracy bounds on horizontal and vertical positions, respectively. Each parameter is defined as the radius of a circle, centered on the reported position, such that the probability of the actual position being outside the circle is 0.05. NAC_P is an index to EPU and VEPU as defined by Table 3-6 in the ASA MASPS (Reference [1]).

NAC_V : NAC_V is an index to the 95% accuracy category of the least accurate rate component as defined in Table 3-7 of ASA MASPS (Reference [1]).

Kalman filter based estimators (e.g., those used by Sensis, STARS, MicroEARTS) provide a measure of error statistics through their covariance matrices. A key assumption in this paper is that the horizontal random error of the position and velocity estimates derived by the TIS-B ground surveillance processor is accurately represented by a bivariate Gaussian process. The uncertainty of position and rate estimates derived from TIS-B surveillance sensors are often not spherical but ellipsoidal in space (elliptical in the horizontal plane) and rotated with respect to the reference coordinate frame. Because NAC is referenced to a *circular* bound that corresponds to the 95% uncertainty, a conversion from an elliptical to circular bound has to take place.

The main result of this paper is an algorithm that exactly computes the 95% containment radius during the steady state phase of flight, two algorithms that approximate the 95% radius, and a comparison between the three methods. A follow-up paper addresses the accuracy during the maneuver phase of the flight.

An ellipse describes the 95% uncertainty region for a Gaussian process in the horizontal plane. Because errors are typically coupled between the components for any chosen reference coordinate frame at any given time, the ellipse will appear rotated, thus complicating the calculation of the major and minor axes. The coupling of errors between coordinate components is manifested in a covariance matrix that has non-zero off-diagonal components. Through a change of basis the covariance matrix can be diagonalized where the diagonal terms provide the squares of the major and minor axes of the corresponding 1- σ error ellipse, and where σ refers to the standard deviation of the Gaussian error. To convert the 1- σ error ellipse to a 95% error ellipse, a conversion factor for the bivariate Gaussian process is computed. Finally, to convert the 95% error ellipse to a 95% error circle, two methods are examined, denoted by *bounded* and *exact*. For the bounded method, two techniques are examined. The techniques consist of basing the containment radius on (a) the root-sum-square (RSS) of the major and minor axes, or (b) only the major axis of the underlying error ellipse (circumscribed technique). It is shown that this method results in an uncertainty region that deviates from the 95% error containment threshold (up to 3 %), where the deviation is proportional to the eccentricity of the underlying error ellipse. The exact method computes a containment radius that maintains the 95% error containment regardless of the eccentricity.

Algorithm Development

In this section, the exact and bounded methods for computing the 95% uncertainty radius (to which NAC is an index) are developed and compared. Both methods derive a circular uncertainty bound from information provided by the covariance matrix that is computed in a Kalman based estimator. The exact method uses variable conversion factors to convert from the 1- σ ellipse to a containment radius

that maintains a 95% containment goal (as defined by the NAC parameter). Algorithms are provided to implement the exact method. A real-time computation of the ratio of the elliptical axes regardless of the rotation of the ellipse with respect to the reference coordinate frame is provided, as well as off-line conversion factors that are captured as a look-up table or an equation as a function of the major-to-minor axes ratio. The bounded method uses a fixed conversion factor to arrive at a containment radius from the 1- σ ellipse.

Definition of the Covariance Matrix

A Kalman based estimator provides a measure of the estimation accuracy through a *covariance matrix*. The covariance matrix is defined in terms of the zero-mean Gaussian estimation vector, \mathbf{E}_k , that is the difference between the measured and predicted quantities (also called the residual vector).

Let

\mathbf{E}_k = 4 x1 vector of the horizontal random error (position and rate) between the state and the “best” estimate of the state based on all measurements at t_0, t_1, \dots, t_k .

$\boldsymbol{\mu}_k$ = 4x1 vector of the mean for the horizontal error (it is set to $\mathbf{0}$ for relative accuracy and non-zero for absolute accuracy).

$\text{Cov}[\mathbf{E}_k]$ = 4 x 4 error covariance matrix (linearized in the case of an Extended Kalman Filter).

$$\mathbf{E}_k = [\Delta x \quad \Delta \dot{x} \quad \Delta y \quad \Delta \dot{y}]$$

(1)

where $(\Delta x, \Delta y)$ are the position and $(\Delta \dot{x}, \Delta \dot{y})$ are the rate errors in the horizontal plane.

$$\mathbf{P} = \text{Cov}[\mathbf{E}_k] = E[\mathbf{E}_k \mathbf{E}_k^T]$$

$$= \begin{bmatrix} E[\Delta x \Delta x] & E[\Delta x \Delta \dot{x}] & E[\Delta x \Delta y] & E[\Delta x \Delta \dot{y}] \\ E[\Delta \dot{x} \Delta x] & E[\Delta \dot{x} \Delta \dot{x}] & E[\Delta \dot{x} \Delta y] & E[\Delta \dot{x} \Delta \dot{y}] \\ E[\Delta y \Delta x] & E[\Delta y \Delta \dot{x}] & E[\Delta y \Delta y] & E[\Delta y \Delta \dot{y}] \\ E[\Delta \dot{y} \Delta x] & E[\Delta \dot{y} \Delta \dot{x}] & E[\Delta \dot{y} \Delta y] & E[\Delta \dot{y} \Delta \dot{y}] \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_x^2 & \sigma_{x\dot{x}}^2 & \sigma_{xy}^2 & \sigma_{x\dot{y}}^2 \\ \sigma_{x\dot{x}}^2 & \sigma_{\dot{x}}^2 & \sigma_{\dot{x}y}^2 & \sigma_{\dot{x}\dot{y}}^2 \\ \sigma_{xy}^2 & \sigma_{\dot{x}y}^2 & \sigma_y^2 & \sigma_{y\dot{y}}^2 \\ \sigma_{x\dot{y}}^2 & \sigma_{\dot{x}\dot{y}}^2 & \sigma_{y\dot{y}}^2 & \sigma_{\dot{y}}^2 \end{bmatrix}$$

(2)

where “^T” denotes the transpose of a matrix.

Justification for Partitioning the Covariance Matrix

The covariance matrix can be partitioned into position and rate terms such that for position terms we can define the following submatrices:

$$\mathbf{E}_{pos} = [\Delta x \ \Delta y] \quad (3)$$

$$\begin{aligned} \mathbf{P}_{pos} &= \text{Cov}[\mathbf{E}_{pos}] = E[\mathbf{E}_{pos} \mathbf{E}_{pos}^T] \\ &= \begin{bmatrix} E[\Delta x \ \Delta x] & E[\Delta x \ \Delta y] \\ E[\Delta y \ \Delta x] & E[\Delta y \ \Delta y] \end{bmatrix} = \begin{bmatrix} \sigma_x^2 & \sigma_{xy}^2 \\ \sigma_{xy}^2 & \sigma_y^2 \end{bmatrix} \end{aligned} \quad (4)$$

The justification for partitioning lies in a property of the multivariate Gaussian distribution (Reference [2]):

Theorem: Let \mathbf{E}_1 be any k components of a multivariate Gaussian vector \mathbf{E} with mean vector $\boldsymbol{\mu}$ and covariance \mathbf{P} . Then, \mathbf{E}_1 is itself a multivariate Gaussian vector with mean vector $\boldsymbol{\mu}_1$ and covariance matrix \mathbf{P}_{11} , where $\boldsymbol{\mu}_1$ and \mathbf{P}_{11} are appropriately constructed from $\boldsymbol{\mu}$ and \mathbf{P} .

Similarly, for rate terms we can define

$$\mathbf{E}_{rate} = [\Delta \dot{x} \ \Delta \dot{y}] \quad (5)$$

$$\begin{aligned} \mathbf{P}_{rate} &= \text{Cov}[\mathbf{E}_{rate}] = E[\mathbf{E}_{rate} \mathbf{E}_{rate}^T] \\ &= \begin{bmatrix} E[\Delta \dot{x} \ \Delta \dot{x}] & E[\Delta \dot{x} \ \Delta \dot{y}] \\ E[\Delta \dot{y} \ \Delta \dot{x}] & E[\Delta \dot{y} \ \Delta \dot{y}] \end{bmatrix} = \begin{bmatrix} \sigma_{\dot{x}}^2 & \sigma_{\dot{x}\dot{y}}^2 \\ \sigma_{\dot{x}\dot{y}}^2 & \sigma_{\dot{y}}^2 \end{bmatrix} \end{aligned} \quad (6)$$

Now the problem of estimating the uncertainty region for position and rate can be analyzed separately.

Derivation of the 1- σ Uncertainty Ellipse from the Covariance Matrix and Calculation of its Major and Minor Axes

This section shows how a 1- σ error uncertainty ellipse that may be rotated with respect to the reference coordinate frame is derived from the covariance matrix. It derives an algorithm for computing the major and minor axes of the 1- σ

error ellipse regardless of the orientation of the ellipse.

The bivariate Gaussian density of \mathbf{E}_{pos} and \mathbf{E}_{rate} may now be written as

$$\begin{aligned} f_{\mathbf{E}_{pos}}(\mathbf{e}_{pos}) &= \frac{(\det \mathbf{P}_{pos})^{-1}}{2\pi} \exp(-1/2(\mathbf{e}_{pos}^T \mathbf{P}_{pos}^{-1} \mathbf{e}_{pos})) \\ &= \frac{(\det \mathbf{P}_{pos})^{-1}}{2\pi} \exp(-a_{pos}/2) \end{aligned} \quad (7)$$

$$\begin{aligned} f_{\mathbf{E}_{rate}}(\mathbf{e}_{rate}) &= \frac{(\det \mathbf{P}_{rate})^{-1}}{2\pi} \exp(-1/2(\mathbf{e}_{rate}^T \mathbf{P}_{rate}^{-1} \mathbf{e}_{rate})) \\ &= \frac{(\det \mathbf{P}_{rate})^{-1}}{2\pi} \exp(-a_{rate}/2) \end{aligned} \quad (8)$$

Standard notation is used here, that is upper case \mathbf{E} refers to the random error vector, and lower case \mathbf{e} refers to the independent variable (a vector) of the density function. The density $f_{\mathbf{E}}(\mathbf{e}_{pos})$ will be constant over the plane for values of $(\Delta x, \Delta y)$ such that the exponent, a_{pos} , is fixed. These values of Δx and Δy define the points on an ellipse in the $(\Delta x, \Delta y)$ plane (Reference [2]).

$$\begin{aligned} a_{pos} &= \mathbf{e}_{pos}^T \mathbf{P}_{pos}^{-1} \mathbf{e}_{pos} \\ &= [\Delta x \ \Delta y] \begin{bmatrix} \sigma_x^2 & \sigma_{xy}^2 \\ \sigma_{xy}^2 & \sigma_y^2 \end{bmatrix}^{-1} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \\ &= \frac{1}{1-\rho^2} \left(\frac{\Delta x^2}{\sigma_x^2} - 2\rho \frac{\Delta x \Delta y}{\sigma_x \sigma_y} + \frac{\Delta y^2}{\sigma_y^2} \right) \end{aligned}$$

where

$$\rho = \frac{\sigma_{xy}^2}{\sigma_x \sigma_y} \quad (9)$$

The major axis of the ellipse is at angle θ to the Δx axis (see Figure 1), calculated as

$$\theta = \frac{1}{2} \arctan \left(\frac{2\rho \sigma_x \sigma_y}{\sigma_x^2 - \sigma_y^2} \right) \quad (10)$$

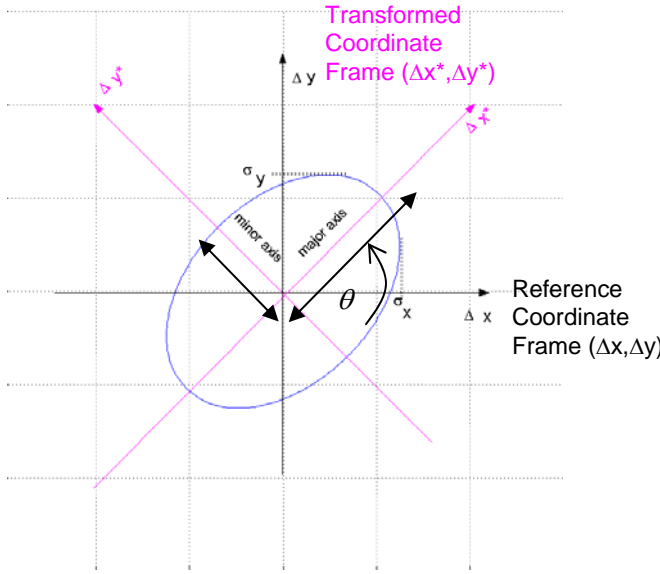


Figure 1. Ellipse Represented in Reference and Transformed Coordinate Frames

Similarly, the density $f_{\mathbf{E}}(\mathbf{e}_{rate})$ will be constant over the plane for values of $(\Delta\dot{x}, \Delta\dot{y})$ such that the exponent, a_{rate} , is fixed. These values of $\Delta\dot{x}$ and $\Delta\dot{y}$ define the points on an ellipse in the $(\Delta\dot{x}, \Delta\dot{y})$ plane.

$$a_{rate} = \mathbf{e}_{rate}^T \mathbf{P}_{rate}^{-1} \mathbf{e}_{rate} = \frac{1}{1 - \rho_{rate}^2} \left(\frac{\Delta\dot{x}^2}{\sigma_{\dot{x}}^2} - 2\rho_{rate} \frac{\Delta\dot{x}\Delta\dot{y}}{\sigma_{\dot{x}}\sigma_{\dot{y}}} + \frac{\Delta\dot{y}^2}{\sigma_{\dot{y}}^2} \right) \quad (11)$$

where

$$\rho_{rate} = \frac{\sigma_{\dot{x}\dot{y}}^2}{\sigma_{\dot{x}}\sigma_{\dot{y}}}$$

The major axis of the ellipse is at angle

$$\theta = \frac{1}{2} \arctan \left(\frac{2\rho_{rate} \dot{\sigma}_x \dot{\sigma}_y}{\dot{\sigma}_x^2 - \dot{\sigma}_y^2} \right) \quad (12)$$

to the $\Delta\dot{x}$ axis.

Equations (9) and (11) present ellipses that are rotated by an angle θ . The covariance \mathbf{P} is a nonsingular matrix, so there exists a similarity transformation \mathbf{A} such that

$$\mathbf{P} = \mathbf{A} \mathbf{\Lambda} \mathbf{A}^{-1} \quad (13)$$

where \mathbf{A} is formed from the eigenvectors of \mathbf{P} associated respectively with the eigenvalues λ_i , where λ_i is the (i,i) element of the diagonal matrix $\mathbf{\Lambda}$.

Substituting (13) into (9) and (11) yields

$$\begin{aligned} a &= \mathbf{e}_{pos}^T \mathbf{P}_{pos}^{-1} \mathbf{e}_{pos} \\ &= \mathbf{e}_{pos}^T (\mathbf{A} \mathbf{\Lambda}_{pos} \mathbf{A}^T)^{-1} \mathbf{e}_{pos} \\ &= (\mathbf{A}^T \mathbf{e}_{pos})^T \mathbf{\Lambda}_{pos}^{-1} (\mathbf{A}^T \mathbf{e}_{pos}) \\ &= \mathbf{e}_{pos,N}^T \mathbf{\Lambda}_{pos}^{-1} \mathbf{e}_{pos,N} \\ &= [\Delta x_N \quad \Delta y_N] \begin{bmatrix} \lambda_{pos,1} & 0 \\ 0 & \lambda_{pos,2} \end{bmatrix}^{-1} \begin{bmatrix} \Delta x_N \\ \Delta y_N \end{bmatrix} \\ &= \left(\frac{\Delta x_N^2}{\lambda_{pos,1}} + \frac{\Delta y_N^2}{\lambda_{pos,2}} \right) \end{aligned} \quad (14)$$

$$\begin{aligned} a &= \mathbf{e}_{rate}^T \mathbf{P}_{rate}^{-1} \mathbf{e}_{rate} \\ &= \mathbf{e}_{rate}^T (\mathbf{A} \mathbf{\Lambda}_{rate} \mathbf{A}^T)^{-1} \mathbf{e}_{rate} \\ &= (\mathbf{e}_{rate})^T \mathbf{\Lambda}_{rate}^{-1} (\mathbf{e}_{rate} \mathbf{A}) \\ &= \mathbf{e}_{rate,N}^T \mathbf{\Lambda}_{rate}^{-1} \mathbf{e}_{rate,N} \\ &= [\Delta\dot{x}_N \quad \Delta\dot{y}_N] \begin{bmatrix} \lambda_{rate,1} & 0 \\ 0 & \lambda_{rate,2} \end{bmatrix}^{-1} \begin{bmatrix} \Delta\dot{x}_N \\ \Delta\dot{y}_N \end{bmatrix} \\ &= \left(\frac{\Delta\dot{x}_N^2}{\lambda_{rate,1}} + \frac{\Delta\dot{y}_N^2}{\lambda_{rate,2}} \right) \end{aligned} \quad (15)$$

where $\Delta x_N, \Delta y_N, \Delta\dot{x}_N, \Delta\dot{y}_N$ are components expressed in transformed coordinate frames (for position and rate error) for which the error ellipses are orthogonal. In Equations (14) and (15) the relationship

$$\mathbf{e}_{m,N} = \mathbf{A}^T \mathbf{e}_m, \text{ where } m = \text{pos or rate} \quad (16)$$

describes a change of basis in rotation only, thus $\det(\mathbf{A}) = 1$ or -1 and $\mathbf{A}^{-1} = \mathbf{A}^T$.

It can be seen from Equations (14) and (15) that for $a = 1$, the transformed ellipse has the familiar expression of an ellipse whose minor and major axes are aligned with the coordinate frame:

$$\frac{x^2}{axis_{major}^2} + \frac{y^2}{axis_{minor}^2} = 1 \quad (17)$$

The relations between $axis_{major}$ and $axis_{minor}$ are illustrated in Figure 1. Comparing Equation (17) to Equations (14) or (15), it is apparent that the eigenvalues, λ_1 and λ_2 , are the squares of the major and minor axes.

The eigenvalues, λ_1 and λ_2 , of a symmetric 2x2 matrix

$$P = \begin{bmatrix} \sigma_x^2 & \sigma_{xy}^2 \\ \sigma_{xy}^2 & \sigma_y^2 \end{bmatrix} \quad (18)$$

are given by the equation:

$$\det \left(\begin{bmatrix} \sigma_x^2 & \sigma_{xy}^2 \\ \sigma_{xy}^2 & \sigma_y^2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) = 0$$

$$\lambda_{1,2} = \frac{(\sigma_x^2 + \sigma_y^2) \pm \sqrt{(\sigma_x^2 - \sigma_y^2)^2 + 4\sigma_{xy}^4}}{2} \quad (19)$$

$$"1-\sigma" \text{ axis}_{major} = \max(+\sqrt{\lambda_1}, +\sqrt{\lambda_2})$$

$$"1-\sigma" \text{ axis}_{minor} = \min(+\sqrt{\lambda_1}, +\sqrt{\lambda_2})$$

Conversion of the 1- σ Ellipse to the 95% Ellipse

In this section, the scaling of the axes of the 1- σ error ellipse (computed from the eigenvalues) to achieve the 95% error ellipse is derived. First, beginning with Equation (20), the probability density (rotated for independence) is integrated over the ellipse bounded by $k\sigma_x$ and $k\sigma_y$ to calculate the desired probability, P .

$$P = \iint_{\left\{x,y: \frac{x^2}{(k\sigma_x)^2} + \frac{y^2}{(k\sigma_y)^2} < 1\right\}} \frac{1}{2\pi\sigma_x\sigma_y} e^{-\left(\frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2}\right)} dx dy \quad (20)$$

Next apply a change of variable. Let

$$u = \frac{x}{\sigma_x}, \quad v = \frac{y}{\sigma_y}, \text{ then (20) becomes}$$

$$P = \iint_{\left\{u,v: u^2 + v^2 < k^2\right\}} \frac{1}{2\pi} e^{-\left(\frac{u^2}{2} + \frac{v^2}{2}\right)} dudv \quad (21)$$

Next, apply a change of variable. Let $u = r \cos \theta$, $v = r \sin \theta$, and $dudv = r dr d\theta$, then

$$P = \int_0^k \int_0^{2\pi} \frac{1}{2\pi} e^{-\left(\frac{r^2}{2}\right)} r d\theta dr = 1 - e^{-\frac{k^2}{2}} \quad (22)$$

Solving (22) for k yields the desired conversion factor.

$$k = \sqrt{-2 \ln(1 - P)} \quad (23)$$

$$\Rightarrow \text{for } P = 0.95, k = 2.4477 \quad (24)$$

As shown in Equation (22), the conversion factor of 2.4477 applied to both axes will convert the 1- σ to the 95% error ellipse when the bivariate Gaussian density function is aligned with the coordinate frame.

Derivation of Methods for Computing 95% Uncertainty Radius from the 1- σ Uncertainty Ellipse

In this section, the exact and bounded methods for converting the 1- σ uncertainty ellipse to a 95% uncertainty radius are derived and compared.

Exact Method

The probability of containing the error over a *circular* region is given by Equation (25), where, as before, the coordinates are rotated for independence. Note the difference between Equations (25) and (20) is in the limits of integration, i.e., in Equation (20) the limits are defined by a circle of radius $k\sigma_y$. Unlike Equation (20), the solution to Equation (25) cannot be found in a closed form and requires the numerical integration of the probability density function derived next.

$$P = \iint_{\left\{x, y : \frac{x^2}{(k\sigma_y)^2} + \frac{y^2}{(\sigma_x)^2} < 1\right\}} \frac{1}{2\pi\sigma_x\sigma_y} e^{-\left(\frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2}\right)} dx dy$$

Let $u = \frac{x}{\sigma_x}$, $v = \frac{y}{\sigma_y}$, then

$$(25)$$

$$P = \iint_{\left\{u, v : \frac{u^2}{\sigma_y^2} + \frac{v^2}{\sigma_x^2} < 1\right\}} \frac{1}{2\pi} e^{-\left(\frac{u^2}{2} + \frac{v^2}{2}\right)} du dv$$

$$(26)$$

Next, apply a change of variables $u = r \cos \theta$, $v = r \sin \theta$, and $du dv = r dr d\theta$, then the area of integration, A , is given by

$$A = \frac{r^2 \cos^2 \theta}{k^2 \frac{\sigma_y^2}{\sigma_x^2}} + \frac{r^2 \sin^2 \theta}{k^2} < 1$$

$$(27)$$

Solving for r , (27) becomes

$$r < \frac{k (\sigma_y / \sigma_x)}{\sqrt{\cos^2 \theta + (\sigma_y / \sigma_x) \sin^2 \theta}}$$

$$(28)$$

and substituting into (26) yields

$$P = \int_0^{2\pi} \int_{r < \frac{k (\sigma_y / \sigma_x)}{\sqrt{\cos^2 \theta + (\sigma_y / \sigma_x) \sin^2 \theta}}} \frac{1}{2\pi} e^{-\left(\frac{r^2}{2}\right)} r dr d\theta$$

$$(29)$$

Equation (29) shows that the probability, P is a function of the conversion factor, k , and, more importantly, a function of the ratio σ_y / σ_x (i.e., *not* the individual values for σ_y and σ_x). However, it is

easier to perform the numerical integration via the substitutions shown below. Starting with Equation (25) where

$$P = \iint_{\left\{x, y : x^2 + y^2 < \left((k\sigma_y)^2 + (\sigma_x)^2\right)\right\}} \frac{1}{2\pi\sigma_x\sigma_y} e^{-\left(\frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2}\right)} dx dy$$

$$(30)$$

Apply a change of variables. Let $x = r \cos \theta$, $y = r \sin \theta$, and $dy dx = r dr d\theta$, then (30) becomes

$$P = \int_0^{2\pi} \int_0^{k\sigma_y} \frac{1}{2\pi\sigma_x\sigma_y} e^{-\left(\frac{r^2 \cos^2 \theta}{2\sigma_x^2} + \frac{r^2 \sin^2 \theta}{2\sigma_y^2}\right)} r dr d\theta$$

$$(31)$$

A table, not shown here, was built from Equation (31) as a function of the ratio $\sigma_{major} / \sigma_{minor}$. The top half of Figure 2 depicts the conversion factor, k , as a function of the ratio $\sigma_{major} / \sigma_{minor}$ and shows that for a circular uncertainty region, corresponding to ratio = 1, the conversion factor is 2.4477. As the uncertainty shape becomes more elongated, corresponding to increasing ratio of axes, the conversion factor becomes 1.9625. The bottom half of Figure 2 shows that for the variable conversion factors a stable and constant 95% containment is maintained. Figure 3 shows that the variable conversion factors can be approximated by the following expression:

$$k = \frac{(2.4477 - 1.9625)}{ratio^3} + 1.9625 = \frac{.4852}{ratio^3} + 1.9625$$

$$\text{where } ratio = \frac{\sigma_{major}}{\sigma_{minor}}$$

$$(32)$$

Figure 4 depicts the different error bounds: 1- σ elliptical in red, 95% elliptical in blue, and 95% circular in green.

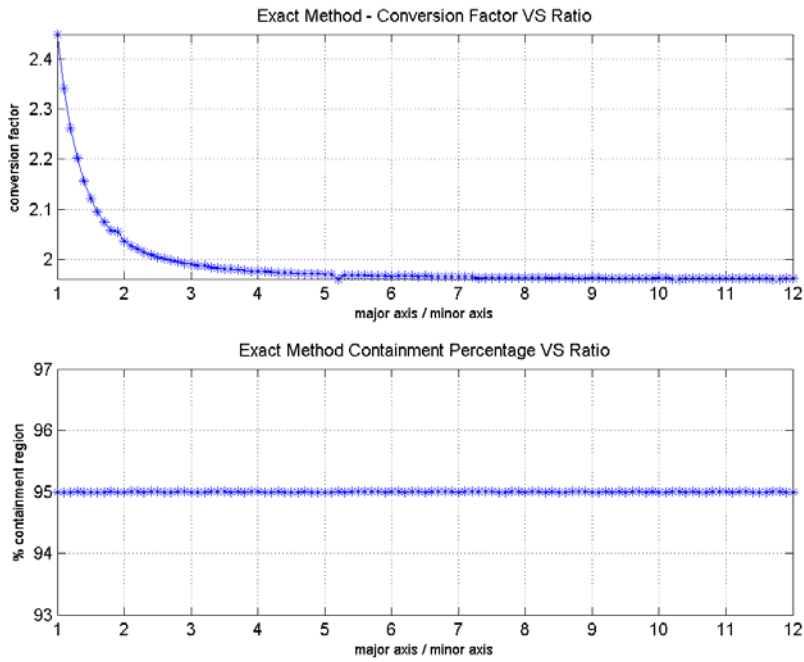


Figure 2. Exact Method: Variable Conversion Factors and Fixed Containment Percentage as a Function of Ratio of Major-to-Minor Ellipse Axes

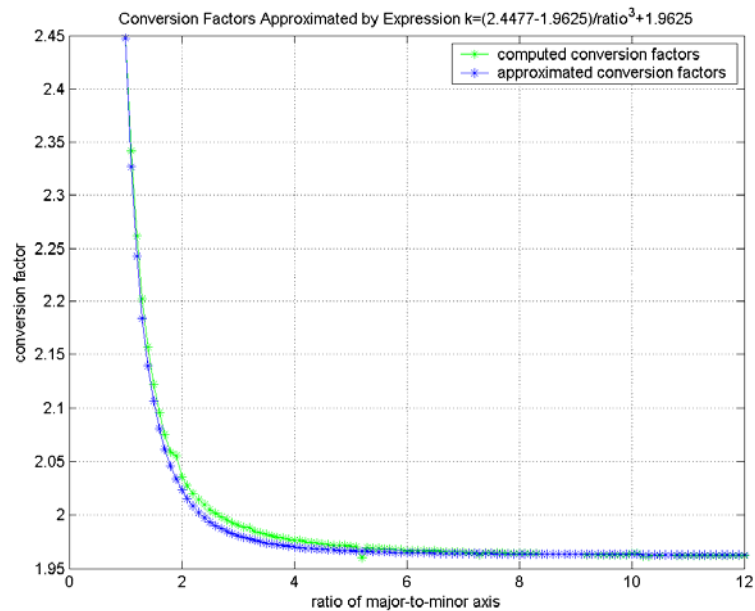


Figure 3. Exact Method: Approximation of the Variable Conversion Factors

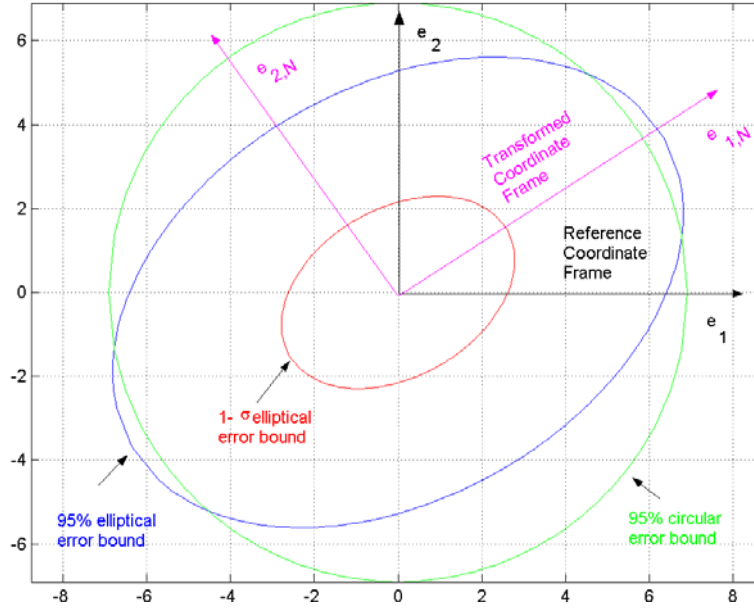


Figure 4. Illustration of Error Bounds: 1- σ Elliptical, 95% Elliptical, and 95% Circular Using the Exact Method

Bounded Method

The bounded method uses a fixed conversion factor to convert from the 1- σ error ellipse to an uncertainty radius. Two sub-techniques of this method, denoted by *RSS* and *circumscribed*, are investigated. An *RSS* radius is computed by applying a fixed conversion factor to the root-sum-squared of the standard deviation of the error in the horizontal plane (i.e., the trace of the covariance submatrix). A circumscribed radius is computed by applying a fixed conversion factor to the major axis of the 1- σ error ellipse. In both cases it will be shown that a fixed conversion factor will result in a circular containment region that deviates from the 95% containment threshold, the deviation being dependent on the choice of conversion factor and elliptical eccentricity, i.e., no single conversion factor will render a 95% threshold over a range of plausible¹ eccentricity values.

¹ The range of ratios is from one to twelve. The upper limit of 12 is chosen to encompass the case of a single long range radar with azimuth accuracy of ± 0.176 deg and range accuracy of $\pm 1/16$ nmi. At a maximum range of 250 nmi, the ratio of axes of the uncertainty ellipse is approximately 12 ($= (250 \text{ [nmi]} * 0.176 \text{ [deg]} * \pi / 180 \text{ [rad/deg]}) / (1/16 \text{ [nmi]})$)

The RSS technique computes a radius of containment as follows:

$$RSS_k = \sqrt{(k * \sigma_x)^2 + (k * \sigma_y)^2} = k \sqrt{\sigma_x^2 + \sigma_y^2} \quad (33)$$

where

σ_x, σ_y are components in the covariance submatrix as defined in Equation (4) and shown in Figure 1.

k is a conversion factor that scales the 1- σ containment up to 95% containment. Note that Equation (33) is the trace of the covariance submatrix scaled by a factor k . In linear algebra, the trace of an n -by- n square matrix \mathbf{P} is defined to be the sum of the elements on the main diagonal (the diagonal from the upper left to the lower right) of \mathbf{P} , i.e.

$$\text{tr}(\mathbf{P}) = \mathbf{P}[1,1] + \mathbf{P}[2,2] + \dots + \mathbf{P}[n,n]. \quad (34)$$

It is interesting to note that calculating the RSS radius does not require finding the major and minor axes of the ellipse (i.e., the square roots of λ_1 and λ_2) and can use the component of a covariance matrix that has not been diagonalized (i.e., σ_x and σ_y). This is explained as follows. Recall that the

transformation from the rotated ellipse to the orthogonal ellipse shown earlier involves the operation $\mathbf{P} = \mathbf{A}\mathbf{A}\mathbf{A}^{-1}$ (see Equation 13). Because \mathbf{A} is an invertible matrix (rotation matrices are always invertible), \mathbf{P} and \mathbf{A} are similar, and by the cyclic property of the trace,

$$\text{tr}(\mathbf{P}) = \text{tr}(\mathbf{A}). \quad (35)$$

Thus, Equation (33) is equivalent to

$$RSS_k = \sqrt{(k * \sqrt{\lambda_1})^2 + (k * \sqrt{\lambda_2})^2} = k\sqrt{\lambda_1 + \lambda_2} \quad (36)$$

Although the axes of the ellipse do not have to be known to compute the RSS radius, the containment region derived from the RSS radius will vary for any chosen (fixed) conversion factor, k , depending on the ratio of the ellipse axes. The relationship between the probability of containment and k is derived below in Equation (37). Note that the differences between Equation (37) and previously derived Equations (20) and (25) are in the limits of integration. Equation (20) integrates over an ellipse and has an analytic solution. Equation (25) integrates over a circle of radius $k\sigma_y$ and requires numerical integration because it does not have an analytical solution. Equation (37) integrates over a circle of radius $k(\sigma_x^2 + \sigma_y^2)^{0.5}$ and requires numerical integration.

$$P = \iint_{\left\{x,y: x^2 + y^2 < \left((k\sigma_y)^2 + (k\sigma_x)^2\right)\right\}} \frac{1}{2\pi\sigma_x\sigma_y} e^{-\left(\frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2}\right)} dx dy$$

Let $u = \frac{x}{\sigma_x}$, $v = \frac{y}{\sigma_y}$, then

(37)

$$P = \iint_{\left\{u,v: \sigma_x^2 u^2 + \sigma_y^2 v^2 < k^2(\sigma_y^2 + \sigma_x^2)\right\}} \frac{1}{2\pi} e^{-\left(\frac{u^2}{2} + \frac{v^2}{2}\right)} dudv \quad (38)$$

Next, apply a change of variables. Let $u = r \cos \theta$, $v = r \sin \theta$, and $dudv = r dr d\theta$, then the area of integration, A , is given by

$$A = r^2 \sigma_x^2 \cos^2 \theta + r^2 \sigma_y^2 \sin^2 \theta < k^2(\sigma_y^2 + \sigma_x^2) \quad (39)$$

Solving for r in (39) yields

$$r < \frac{k(\sigma_y^2 + \sigma_x^2)^{0.5}}{\sqrt{\sigma_x^2 \cos^2 \theta + \sigma_y^2 \sin^2 \theta}} = \frac{k\left(\left(\sigma_y/\sigma_x\right)^2 + 1\right)^{0.5}}{\sqrt{\cos^2 \theta + (\sigma_y/\sigma_x)^2 \sin^2 \theta}} \quad (40)$$

After the change of variables, Equation (40) becomes

$$P = \int_0^{2\pi} \frac{1}{2\pi} e^{-\left(\frac{r^2}{2}\right)} r dr d\theta = 1 - e^{-\frac{k^2}{2}} \quad (41)$$

Equation (41) shows that the probability, P is a function of the conversion factor, k , and, more importantly, a function of the ratio σ_y/σ_x (i.e., *not* the individual values for σ_y and σ_x). However, it is easier to perform the numerical integration via the substitutions shown below:

$$P = \iint_{\left\{x,y: x^2 + y^2 < k^2(\sigma_x^2 + \sigma_y^2)\right\}} \frac{1}{2\pi\sigma_x\sigma_y} e^{-\left(\frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2}\right)} dx dy \quad (42)$$

Next, apply a change in variable. Let $x = r \cos \theta$, $y = r \sin \theta$, and $dydx = r dr d\theta$, then

$$P = \int_0^{2\pi} \int_0^{k\sqrt{\sigma_x^2 + \sigma_y^2}} \frac{1}{2\pi\sigma_x\sigma_y} e^{-\left(\frac{r^2 \cos^2 \theta}{2\sigma_x^2} + \frac{r^2 \sin^2 \theta}{2\sigma_y^2}\right)} r dr d\theta \quad (43)$$

Figure 5 depicts the different error bounds: 1- σ elliptical in red, 95% elliptical in blue, and RSS circular in green. Figures 6 and 7 illustrate the containment region as a function of the ratio of major-to-minor ellipse axes for two conversion factors, $k = 2.4477$, 1.9625 . The choice of these two

values was taken from the results of an earlier section, where it was shown that to achieve a 95% circular containment these two thresholds bounded the required value of k (from circular to highly elongated ellipse for the underlying uncertainty). Comparing Figures 6 and 7 indicates that the choice of $k = 1.9625$ achieves better performance than $k = 2.4477$. For $k = 1.9625$ the containment region converges to 95% as the ratio increases and exceeds the threshold almost 3% when the $ratio = 1$, whereas for $k = 2.4477$ the containment region only approaches 98.6% as the ratio increases and

exceeds the threshold almost 5% when $the\ ratio = 1$.

For the circumscribed bounded technique, the circular bound is set to the major axis of the $1-\sigma$ error ellipse scaled by a fixed factor. The performance for $k = 1.9625$ was unacceptable because the containment region fell below the 95% for low values of the ratio (i.e., $ratio < 5$). For $k = 2.4477$ the 95% error threshold is met exactly at $ratio = 1$, but quickly converges to 98.5% (i.e., for $ratio \geq 3$), as shown in Figure 8.

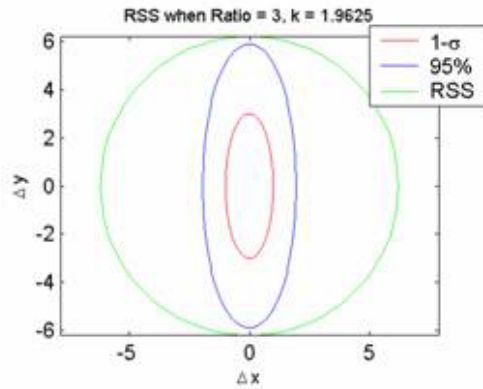


Figure 5. Example of Error Bounds: $1-\sigma$ Elliptical, 95% Elliptical, and Circular Using the RSS Method

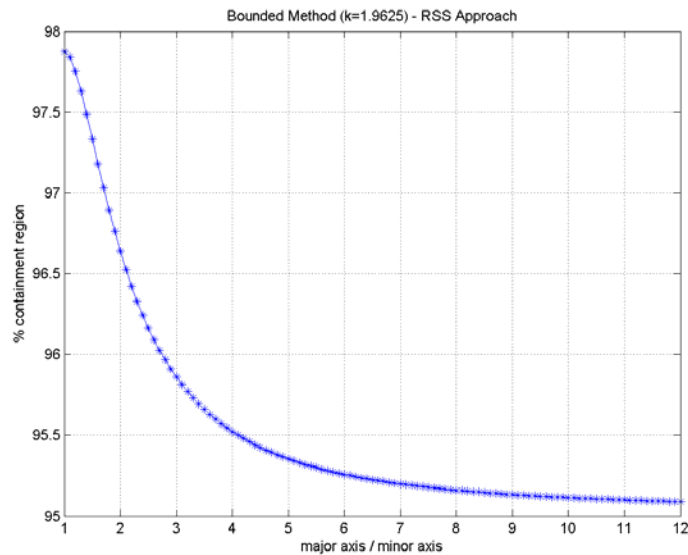


Figure 6. RSS Technique: Containment Region as a Function of Ratio of Major-to-Minor Ellipse Axes Using a Fixed Conversion Factor =1.9625

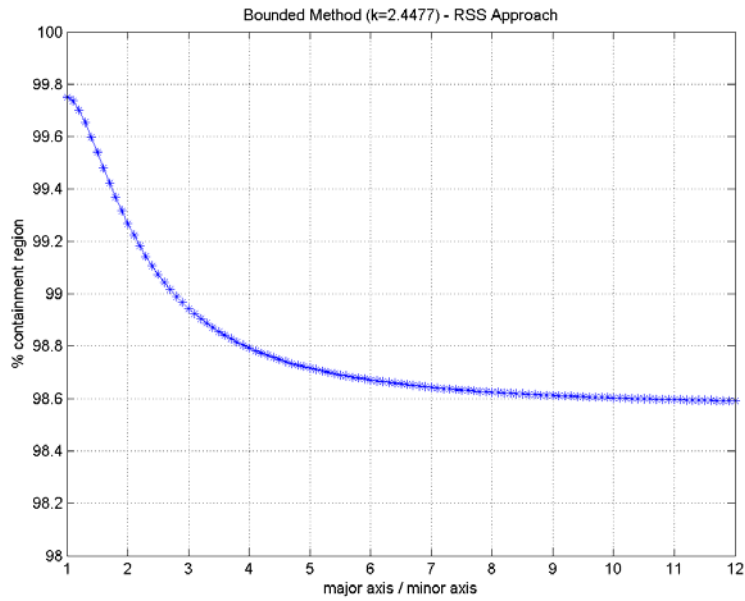


Figure 7. RSS Technique: Containment Region as a Function of Ratio of Major-to-Minor Axes Using a Fixed Conversion Factor = 2.4477

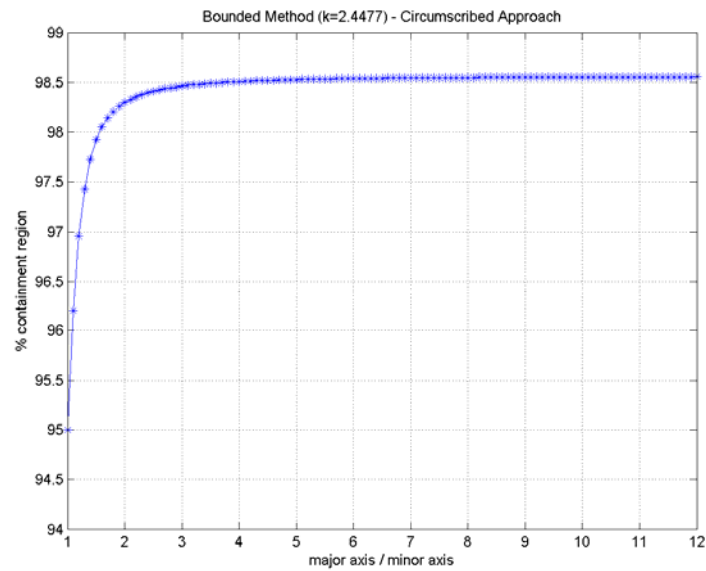


Figure 8. Circumscribed Technique: Containment Region as a Function of Ratio of Major-to-Minor Axes Using a Fixed Conversion Factor = 2.4477

Comparison of Methods

The exact method maintains a 95% containment threshold for all ratios. The real time processing is moderate and consists of solving for the eigenvalues and either storing a lookup table or computing an approximation (see Equation (32)). The circumscribed bounded technique also requires the eigenvalues to compute the major axis and provides the most conservative estimate for the containment bound. The RSS bounded technique is the simplest of the three techniques and provides a less conservative estimate of the containment bound than the circumscribed technique.

Algorithm Summary

The algorithm for computing NAC for position and rate using the *exact* method can be summarized in three simple steps

- Partition the covariance matrix into position-only and rate-only submatrices.
- Find the eigenvalues of each submatrix. The positive square roots of the eigenvalues correspond to the major and minor axes of the 1- σ error ellipse in a rotated coordinate frame that is aligned with the axes.
- Compute the ratio of the major to minor axes. Use it as an index to a look-up table of conversion factors that have been computed off-line, or approximate the conversion factor from an expression [Equation (32)] that is a function of the axes.

The algorithm for the bounded circumscribed technique can be summarized in three steps. The first two steps are the same as the exact method. The third step is

- Multiply the major axis by a constant conversion factor, $k = 2.447$.

The algorithm for the bounded RSS technique can be summarized by three steps. The first step is the same as the other two techniques. The next two steps are [Equation (33)]

- Compute the RSS of the diagonal components of the partitioned matrix
- Multiply by constant conversion factor, $k = 1.9625$.

Example

Given a 4x4 covariance matrix for the horizontal position and rate where the units of length and time are feet and seconds:

$$P = \begin{bmatrix} 4.84e+5 & d & -3.176e+5 & d \\ d & 7.657e+2 & d & -1.452e+2 \\ -3.176e+5 & d & 6.577e+5 & d \\ d & -1.452e+2 & d & 8.877e+2 \end{bmatrix} \quad (44)$$

where d = don't care, and the format is according to Equation (2), repeated here as Equation (45) for the reader's convenience:

$$\begin{aligned} P &= Cov[E_k] = E[E_k E_k^T] \\ &= \begin{bmatrix} E[\Delta x \Delta x] & E[\Delta x \Delta \dot{x}] & E[\Delta x \Delta y] & E[\Delta x \Delta \dot{y}] \\ E[\Delta \dot{x} \Delta x] & E[\Delta \dot{x} \Delta \dot{x}] & E[\Delta \dot{x} \Delta y] & E[\Delta \dot{x} \Delta \dot{y}] \\ E[\Delta y \Delta x] & E[\Delta y \Delta \dot{x}] & E[\Delta y \Delta y] & E[\Delta y \Delta \dot{y}] \\ E[\Delta \dot{y} \Delta x] & E[\Delta \dot{y} \Delta \dot{x}] & E[\Delta \dot{y} \Delta y] & E[\Delta \dot{y} \Delta \dot{y}] \end{bmatrix} \\ &= \begin{bmatrix} \sigma_x^2 & \sigma_{x\dot{x}}^2 & \sigma_{xy}^2 & \sigma_{x\dot{y}}^2 \\ \sigma_{x\dot{x}}^2 & \sigma_{\dot{x}}^2 & \sigma_{\dot{x}y}^2 & \sigma_{\dot{x}\dot{y}}^2 \\ \sigma_{xy}^2 & \sigma_{\dot{x}y}^2 & \sigma_y^2 & \sigma_{y\dot{y}}^2 \\ \sigma_{x\dot{y}}^2 & \sigma_{\dot{x}\dot{y}}^2 & \sigma_{y\dot{y}}^2 & \sigma_{\dot{y}}^2 \end{bmatrix} \end{aligned} \quad (45)$$

Partition the covariance matrix P into the submatrices P_{pos} and P_{rate} where

$$P_{pos} = \begin{bmatrix} 4.84e+5 & -3.176e+5 \\ -3.176e+5 & 6.577e+5 \end{bmatrix} \quad (46)$$

and

$$P_{rate} = \begin{bmatrix} 7.657e+2 & -1.452e+2 \\ -1.452e+2 & 8.877e+2 \end{bmatrix} \quad (47)$$

To compute the eigenvalues of P_{pos}

$$\det \left(\begin{bmatrix} 4.84e+5 & -3.176e+5 \\ -3.176e+5 & 6.577e+5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) = 0 \quad (48)$$

Solving for the eigenvalues yields

$$\lambda_{1,2} = \frac{(4.84e+5 + 6.577e+5) \pm \sqrt{(4.84e+5 - 6.577e+5)^2 + 4(-3.176e+5)^2}}{2} \quad (49)$$

$$\lambda_1 = 9.0011e+5; \quad \lambda_2 = 2.4159e+5 \quad (50)$$

The major and minor of the error ellipse are given by

$$\text{major axis} = \sqrt{\lambda_1} = 958.7417 \text{ feet; minor axis} = \sqrt{\lambda_2} = 491.5172 \text{ feet} \quad (51)$$

The ratio of major-to-minor ellipse axes for the position uncertainty can now be computed

$$\text{ratio} = \frac{958.7417}{491.5172} = 1.95 \quad (52)$$

From Equation (32)

$$k = \frac{0.4852}{1.95^3} + 1.9625 = 2.0279 \quad (53)$$

Thus, the radius of 95% containment is

$$\begin{aligned} \text{radius} &= k * \text{major axis} = 2.0279 * 958.7417 \text{ ft} \\ &= 1940 \text{ ft} = 0.368 \text{ nmi} \end{aligned}$$

From Table 3-6 in ASA MASPS this corresponds to $\text{NAC}_p = 6$.

Similarly, for calculating NAC_v we first compute the major and minor axes of the $1-\sigma$ uncertainty ellipse

$$\det \left(\begin{bmatrix} 7.657e+2 & -1.452e+2 \\ -1.452e+2 & 8.877e+2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) = 0 \quad (54)$$

$$\lambda_{1,2} = \frac{(7.657e+2 + 8.877e+2) \pm \sqrt{(7.657e+2 - 8.877e+2)^2 + 4(-1.452e+2)^2}}{2} \quad (55)$$

$$\lambda_1 = 1214.4; \quad \lambda_2 = 698 \quad (56)$$

$$\text{major axis} = \sqrt{\lambda_1} = 34.8 \text{ ft/s; minor axis} = \sqrt{\lambda_2} = 26.4 \text{ ft/s} \quad (57)$$

The ratio of major to minor axes is

$$\text{ratio} = \frac{34.8}{26.4} = 1.3 \quad (58)$$

From Equation (32) $k = 2.1833$, thus the radius of 95% containment is

$$\begin{aligned} \text{radius} &= k * \text{major axis} = 2.1833 * 34.8 \text{ ft/s} \\ &= 75.98 \text{ ft/s} = 23.15 \text{ m/s} \end{aligned} \quad (59)$$

From Table 3-7 in ASA MASPS this corresponds to $\text{NAC}_v = 0$.

For the bounded circumscribed technique, the containment radius for position is (using results from Equation (51))

$$\begin{aligned} \text{radius} &= k * \text{major axis} = 2.447 * 958.7417 \text{ ft} \\ &= 2.346e+3 \text{ ft} = 0.386 \text{ nmi} \end{aligned} \quad (60)$$

and the containment radius for velocity is (using results from equation (58))

$$\text{radius} = k * \text{major axis} = 2.447 * 34.8 \text{ ft/s} = 26 \text{ m/s} \quad (61)$$

For the bounded RSS technique, the containment radius for position and rate (using the partitioned matrix from Equation (46) and (47))

$$\begin{aligned} \text{radius} &= k \sqrt{\sigma_x^2 + \sigma_y^2} = 1.9625 \sqrt{3.84e+5 + 6.577e+5} \\ &= 2096 \text{ ft} = 0.345 \text{ nmi} \end{aligned} \quad (62)$$

$$\begin{aligned} \text{radius} &= k \sqrt{\sigma_x^2 + \sigma_y^2} = 1.9625 \sqrt{7.657e+2 + 8.877e+2} \\ &= 79.8 \text{ ft/s} = 24.32 \text{ m/s} \end{aligned} \quad (63)$$

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